Chapter-1

Differential Equation

Date: 16/11/22

★ **Taylor's theoram** →

If $f(a+h)$ where a is independent of (h), be a func.ⁿ of (h) Such that it can be expanded ascending powers of (h) and this expansion can be differentiable at no. of times then theoram state that:

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots \quad (1)$$

• **Cor. 1** → Putting $\boxed{a = x}$ in eq. (1)

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

• **Cor. 2** → Putting $\boxed{h = x-a}$, in eq. (1), we get

$$f(x) = f(a) + f'(a)(x-a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

▶ Put $a = 0$ and $h = x$ → it becomes Maclaurin's theoram in eq. (1)

Date :
P. No. :

<u>Que</u> Expand $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$?

<u>Sol^n</u> $\Rightarrow$ $f(x) = \sin x$

$$f(a+h) = f(a+x-a) = f\left(\frac{\pi}{2}+x-\frac{\pi}{2}\right) = f(x)$$

$$f(x) = \sin x \Rightarrow f\left(\frac{\pi}{2}\right) = 1$$

$$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{2}\right) = 0$$

$$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{2}\right) = -1$$

$$f'''(x) = -\cos x \Rightarrow f'''\left(\frac{\pi}{2}\right) = 0$$

$$f^{IV}(x) = \sin x \Rightarrow f^{IV}\left(\frac{\pi}{2}\right) = 1$$

By Taylor's theoram

<u>Cor. 2</u> $f(x) = f(a) + f'(a)(x-a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$

$$\sin x = 1 + \left(x-\frac{\pi}{2}\right)\cdot 0 + \frac{\left(x-\frac{\pi}{2}\right)^2}{2}(-1) + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!}(1) + \ldots$$

$$\therefore \frac{\left(x-\frac{\pi}{2}\right)^3}{3!}\cdot 0$$

$$\boxed{\sin x = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} + \ldots}$$

Date :
P. No. :

Assignment!

Que. Expand $\tan x$ in powers of $\left(x+\frac{\pi}{4}\right)$

Que. $\tan\left(x+\frac{\pi}{4}\right)$ As far as x^4 terms and evalute $\tan(46.5^\circ)$ Singnificant

Ans. $\Rightarrow \tan\left(x+\frac{\pi}{4}\right) = 1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\ldots$ to 4 digits.

Taylor's theoram

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$$

$$\boxed{a=\frac{\pi}{4},\quad h=x}$$

$$f\left(x+\frac{\pi}{4}\right) = \tan\left(x+\frac{\pi}{4}\right)$$

$$f(x) = \tan x$$

$$f(a) = \tan a$$

$$f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

$$f'(x) = \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right)$$

$$f'\left(\frac{\pi}{4}\right) = (2)^2 = \underline{\underline{4}}$$

put $x = 1.5$

$$\tan(46.5^\circ) = 1 + 2(1.5) + 2(1.5)^2 + \frac{8}{3}(1.5)^3 + \ldots$$

$$\tan(46.5) = 1.0537$$

★ Maxima - Minima ~~of~~ of funcⁿ of two variables →

Let $f(x,y)$

★ Working Rule →

(1) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(2) Solve the eq.ⁿ $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, we get x & y.

then,

(2) The Pair of x & y thus obtain stationary value of $f(x,y)$. Let (a,b) be one of these pairs

(3) Find $r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}$, $t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}$, $s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}$ at the point (a,b).

and Calculate $rt - s^2$.

(4) (i) If $rt - s^2$ is positive and r is greater than to 0, then $f(x,y)$ has ~~fun~~ minimum at (a,b).

(ii) If $rt - s^2$ is positive and r is smaller than to 0, then $f(x,y)$ has maximum at (a,b).

(iii) If $rt-s^2$ is negative, then $f(x,y)$ has neither maximum nor minimum at (a,b).

(iv) If $rt-s^2$ is zero, then ~~f(x,y)~~ has case is doubtful.

Que. Discuss the maximum or minimum value of the func^n $f(x,y)$ is equals to $x^3-4xy+2y^2$.

$$f(x,y) = x^3-4xy+2y^2$$
$$f = x^3-4xy+2y^2$$

$$\frac{\partial f}{\partial x} = 3x^2-4y, \quad \frac{\partial f}{\partial y} = -4x+4y$$

$$3x^2-4y=0$$
$$3x^2-4x=0$$
$$x(3x-4)=0$$
$$\boxed{x=0} \quad 3x-4=0$$
$$\boxed{x=\frac{4}{3}}$$

$$-4x+4y=0$$
$$\boxed{4y=4x}$$
$$\boxed{x=y}$$

$$\boxed{y=\frac{4}{3}} \quad \boxed{y=0}$$

Points are $\rightarrow \left(\frac{4}{3},\frac{4}{3}\right)$ & $(0,0)$ are critical points.

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = (6x)_{(0,0)} = 0 \Rightarrow \boxed{r=0}$$

$$t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = (4)_{(0,0)} = 4 \Rightarrow \boxed{t=4}$$

Date :
P. No. :

$$s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right]_{(0,0)} = \left[\frac{\partial}{\partial x}(-4x+4y)\right]_{(0,0)}$$

$$s = (-4)_{(0,0)} = -4 \Rightarrow \boxed{s = -4}$$

Calculate $rt - s^2 = 0 - 16 = -16$ (−ve)

If $rt-s^2$ is (−ve) then $f(x,y)$ is neither max. nor minimum at $(0,0)$

Calculate At $= \left(\frac{4}{3}, \frac{4}{3}\right)$ $\underbrace{r = 8,\ t = 4,\ s = -4}_{\text{Same as phle Jaisa}}$

$rt - s^2 = 8 \times 4 - 16 = 32 - 16$

$rt - s^2 = 16 = +ve \Rightarrow \left\{ \begin{array}{c} rt - s^2 = +ve \\ r = +ve \end{array} \right\}$

Hence, $f(x,y)$ is minimum at $\left(\frac{4}{3}, \frac{4}{3}\right)$.

The minimum Value of $f\left(\frac{4}{3}, \frac{4}{3}\right) = x^3 - 4xy + 2y^2$

$$= \left(\frac{4}{3}\right)^3 - 4\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) + 2\left(\frac{4}{3}\right)^2$$

$$= \frac{-32}{27}.$$

$(-4x+4y)$

$(0,0)$

$ax.$

Faisa

Que. Discuss the maximum and minimum value of the func.^{n}.

(1) $u = x^3+y^3-3axy$

(2) Assignment! $u = x^3y^2(1-x-y)$

$f = x^3+y^3-3axy$

$\frac{\partial f}{\partial x} = 3x^2-3ay$, $\frac{\partial f}{\partial y} = 3y^2-3ax$

$0 = 3x^2-3ay$, $0 = 3y^2-3ax$

$3ay = 3x^2$, $3ax = 3y^2$

$ay = x^2$, $ax = y^2$

$\boxed{a = \frac{x^2}{y}}$

$\frac{x^2}{y} \cdot x = y^2$

$x^3 = y^3$

$\boxed{x = y}$

$r = \left(\frac{\partial^2 f}{\partial x^2}\right) = (6x)_{(a,a)} = 6a$

$t = \left(\frac{\partial^2 f}{\partial y^2}\right) = (6y)_{(a,a)} = 6a$

$s = \left(\frac{\partial^2 f}{\partial x \partial y}\right) = \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right] = \left[\frac{\partial}{\partial x}(3y^2-3ax)\right]$

$\boxed{S = -3a}$

Calculate $rt-s^2 = 6a \times 6a - (-3a)^2 = 36a^2 - 9a^2$
$= 27a^2$

$rt-s^2 = +ve$

Date :
P. No. :

Hence $rt - s^2$ is +ve and r is -ve (or) +ve, according as (a). We have maximum and minimum according as (a) is -ve (or) +ve.

For $(x = y = a)$.

Date : 22/11/22

★ Curvature, Radius of Curvature, Centre of curvature

(1) Curvature (K) :- Let P & Q are two neighbouring point. On a given continuous curve such that arc PQ = δS and arc $A'P = S$ where A' is fixed point on the curve. Again suppose that the tangent at the point P and Q makes angle ψ, $\psi + \delta\psi$ respectively with the x-axis then $\delta\psi$ is called total curvature of the arc PQ.

(2) The ratio $\frac{\delta\psi}{\delta S}$ is called avg. curvature

(3) $\lim\limits_{\delta x \to 0} \frac{\delta\psi}{\delta S}$ or $\frac{\delta\psi}{\delta s}$ is called curvature of the arc PQ.

It is denoted by Kappa (K).

(4) Reciprocal of the curvature is called Radius of curvature. Radius of curvature is denoted by (ρ)

★ Cartesion formula of Radius of Curvature

$$\boxed{(\rho) = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}}$$

Date :
P. No. :

★ Pedal formula for Radius of Curva. :-

~~$\rho = r \frac{dp}{dr}$~~

$$\boxed{\rho = r\left(\frac{dr}{dP}\right)}$$

where $\boxed{p = r \sin\phi}$

★ Centre of Curvature (β) :-

$$\boxed{\beta = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}}$$

and

$$\boxed{\alpha = x - \frac{\left(\frac{dy}{dx}\right)\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}}$$

★ Circle of Curvature ( ) :-

$$\boxed{(x-\alpha)^2 + (y-\beta)^2 = (\rho)^2}$$

- evolute locus of centre of Curv. (α, β) is called evolute of curve.

Date :
P. No. :

Que. Find the Radius of Curvature of the given curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$, at the point $\left(\frac{1}{4},\frac{1}{4}\right)$.

$$\sqrt{x}+\sqrt{y}=\sqrt{a} \quad -(1)$$

diff. w.r.t (x)

$$\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\sqrt{y}}{\sqrt{x}}=-y^{1/2}x^{-1/2}$$

diff w.r.t (x)

$$\frac{d^2y}{dx^2}=+\frac{1}{2}y^{1/2}x^{-3/2}-\frac{1}{2}x^{-1/2}y^{-1/2}\frac{dy}{dx}$$

$$=\frac{1}{2}\frac{\sqrt{y}}{x^{3/2}}-\frac{1}{2\sqrt{x}\cdot\cancel{\sqrt{y}}}\cdot\left(\frac{-\cancel{\sqrt{y}}}{\sqrt{x}}\right)$$

$$=\frac{1}{2}\left[\frac{\sqrt{y}+\sqrt{x}}{x\sqrt{x}}\right]=\frac{1}{2}\frac{\sqrt{a}}{x^{3/2}} \text{ from eq. (1)}$$

Radius of Curvature

$$P(x,y)=\frac{\left[1+\left(\frac{-\sqrt{y}}{\sqrt{x}}\right)^2\right]^{3/2}}{\frac{\sqrt{a}}{2x^{3/2}}}=\frac{2x^{3/2}\left(1+\frac{y}{x}\right)^{3/2}}{\sqrt{a}}$$

$$=\frac{2x^{3/2}\left(\frac{x+y}{x}\right)^{3/2}}{\sqrt{a}}$$

$$\boxed{P(x,y)=\frac{2(x+y)^{3/2}}{\sqrt{a}}}$$

Date :
P. No. :

$$\rho\left(\frac{1}{4},\frac{1}{4}\right) = \frac{2\left(\frac{1}{4}+\frac{1}{4}\right)^{3/2}}{\sqrt{a}} = \frac{2\left(\cancel{2}\times\frac{1}{2\cancel{4}}\right)^{3/2}}{\sqrt{a}}$$

$$\rho\left(\frac{1}{4};\frac{1}{4}\right) = \frac{2\left(\frac{1}{8}\right)^{\frac{1}{2}}}{\sqrt{a}} = \cancel{2}\frac{1}{\cancel{2}\sqrt{2a}}$$

$$\rho\left(\frac{1}{4},\frac{1}{4}\right) = \frac{1}{\sqrt{2a}}$$

Assignment.

Que. Find the Radius of Curva at the point P the given is cycloid $x = a(t + \sin t)$ $y = a(1 - \cos t)$.

Date :
P. No :

Que Find the coordinates of the centre of curvature for the point (x, y) on the Parabola $y^2 = 4ax$ and also find eq.^{n} of the evolute of parabola.

Here $y^2 = 4ax$ — (1)

diff. w.r.t. (x)

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} = \frac{\sqrt{a}}{\sqrt{x}}$$

diff. w.r.t (x)

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{a}}{2}x^{-3/2} = -\frac{\sqrt{a}}{2x^{3/2}} \Rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{-\sqrt{a}}{2x^{3/2}}}$$

At point (x, y) the coordinates of Centre of curvature

$$\beta = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = y + \frac{\left[1 + \left(\frac{\sqrt{a}}{\sqrt{x}}\right)^2\right]}{\frac{-\sqrt{a}}{2x^{3/2}}}$$

$$\beta = y + \left[\left(\frac{x+a}{x}\right) \times \frac{2x^{3/2}}{-\sqrt{a}}\right] = y + \left[\frac{x+a}{-\sqrt{a}} \times 2x^{3/2}\right]$$

$$\beta = y + \frac{(x+a)}{-\sqrt{a}} \times 2x^{1/2}$$

$$\beta = \frac{-\sqrt{a}y + 2x^{3/2} + 2x^{1/2}a}{-\sqrt{a}} = \frac{-\sqrt{a}\sqrt{4ax} + 2x^{3/2} + 2x^{1/2}a}{-\sqrt{a}}$$

$$\beta = -\left(\frac{4ax^{1/2} + 2x^{3/2}}{\sqrt{a}}\right) \qquad \beta = -\left(\frac{2a\sqrt{x} + 2x^{3/2} + 2x^{1/2}a}{\sqrt{a}}\right)$$

Date :
P. No. :

$$\boxed{\beta = -2x^{3/2}a^{-1/2}} \quad \text{---} (3)$$

$$\boxed{\alpha = 3x + 2a} \quad \text{---} (4)$$

Thus the coordinates of centre of curvature is $\left(3x+2a, \; -\frac{2}{\sqrt{a}}x^{3/2}\right)$.

To evolute

$$3x = \alpha - 2a$$

$$\boxed{x = \frac{\alpha - 2a}{3}} \quad \text{---} (5)$$

$$\boxed{\beta = \frac{-2x^{3/2}}{\sqrt{a}}}$$

Both side square

$$\beta^2 = \frac{4x^3}{a}$$

$$\boxed{x^3 = \frac{a\beta^2}{4}} \text{ from eq. } (6)$$

From eq. (5) & (6)

$$\left(\frac{\alpha - 2a}{3}\right)^3 = \frac{a\beta^2}{4}$$

$$27a\beta^2 = 4(\alpha - 2a)^3$$

The eq.^{n} of locus of (α, β)

$$\boxed{27ay^2 = 4(x - 2a)^3}$$